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MINIMALITY OF HOROSPHERICAL FLOWS

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ABSTRACT

If N is the nilpotent constituent of an Iwasawa decomposition of the semi-simple group G (finite center and no compact factors), it is proved that N acts minimally on G/Γ for every uniform lattice $\Gamma \subseteq G$, generalizing theorems of Hedlund and L. Greenberg.

1. Introduction

The purpose of the present note is to prove

THEOREM 1.1. *Let G be a connected semi-simple Lie group with finite center and no nontrivial compact factor. Fix an Iwasawa decomposition, $G = KAN$, for G. If Γ is a discrete cocompact subgroup of G, the flow $(N, G/\Gamma)$ is minimal.*

Recall that an Iwasawa decomposition entails closed subgroups K, A, and N of G, where K is maximal compact, A is a vector group, and N is a simply connected nilpotent group. AN is a solvable group with commutator subgroup N. Further, there are functions $k: G \to K$, $a: G \to A$, and $n: G \to N$ such that for each $g \in G$, $g = k(g)a(g)n(g)$, and $g \to (k(g), a(g), n(g))$ gives a diffeomorphism between G and $K \times A \times N$.

A flow is *minimal* if each orbit is dense. In the case $G = SL(2, \mathbf{R})$ the natural choice of N is $N = \{\left(\begin{smallmatrix}1 & b\\ 0 & 1\end{smallmatrix}\right) \mid b \in \mathbf{R}\}$, and our theorem reduces to the classical theorem of Hedlund [6] on the minimality of the horocycle flow. Our approach to Theorem 1.1 is motivated by [6] and by Greenberg's paper [5].

We remark that little seems to be known about the following question: for which subgroups H of G is it the case that $(H, G/\Gamma)$ is minimal whenever Γ is discrete and cocompact? In particular, it does not seem to be known whether G contains a one parameter subgroup H with this property. (If N is one dimensional, G is locally isomorphic to $SL(2, \mathbf{R})$.)

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2. The flow $(G, G/N)$

Let H_1, H_2 be closed subgroups of G. If $x \in G$, the point $xH_2 \in G/H_2$ has a dense orbit under H_1 if and only if H_1xH_2 is dense in G. In other words, xH_2 has a dense H_1 orbit if and only if H_1x has a dense H_2 orbit (in $H_1\backslash G$). Thus, $(H_1, G/H_2)$ is minimal if and only if $(H_2, G/H_1)$ is minimal. This is the version for flows of Moore's duality principle [8], [3]. We will make use of it in two ways. First, if H_1, H_2 are such that there exist dense H_1 orbits in G/H_2, then there will also exist dense H_2 orbits in G/H_1. Secondly, to prove Theorem 1.1 it is sufficient to prove that $(\Gamma, G/N)$ is minimal whenever Γ is discrete and cocompact.

Because of the Iwasawa decomposition the homogeneous space G/AN can be identified with K. The action of G on G/AN goes over to the action $T_g: K \to K$, $g \in G$, defined by

$$T_gk = k(gk) \tag{2.1}$$

where $k: G \to K$ is as in Section 1. Restricting the action to Γ, there is also a natural flow (Γ, K). By Moore's ergodicity theorem [8] and our assumption that G has no nontrivial compact factor, the flow $(AN, G/\Gamma)$ has almost all orbits dense. Therefore (Γ, K) has at least one dense orbit. This will be seen below to imply that every orbit is dense.

Let $M = \{m \in K \mid ma = am,\ a \in A\}$ be the centralizer of A in K. M is a closed subgroup of K, and $P = MAN$ is a closed subgroup of G in which N is normal. For this reason we have $k(gm) = k(g)m$, $g \in G$, $m \in M$. Therefore the action (2.1) of G on K and the action $k \to km$ of M on K commute with one another. There is a natural flow $(G, K/M)$ on the quotient space, and for Γ discrete and cocompact it is known that $(\Gamma, K/M)$ is minimal ([2]; see also [4]).

Let $k \in K$ be a point with a dense Γ-orbit. Because the actions of Γ and M commute on K, each point km, $m \in M$ has a dense Γ orbit. If $k' \in K$, then because $(\Gamma, K/M)$ is minimal and K is compact, there exists some $m \in M$ such that km is in the Γ orbit closure of k'. Since km has a dense Γ orbit, k' must have a dense Γ orbit. That is, (Γ, K) is minimal.

Remark. If Γ is discrete, and if G/Γ has finite volume, then $(\Gamma, K/M)$ (and (Γ, K)) are minimal. (Mostow [10].)

Using the Iwasawa decomposition once more we identify G/N with $K \times A$. Define $\alpha(\cdot,\cdot)$: $G \times K \to A$ by $\alpha(g,k) = a(gk)$, and notice that

$$\alpha(g_1g_2, k) = \alpha(g_1, g_2k)\,\alpha(g_2, k). \tag{2.2}$$

It follows that the action T_g: $K \times A \to K \times A$ defined by

(2.3) $$T_g(k,a) = (T_gk, \alpha(g,k)a)$$

satisfies $T_{g_1}T_{g_2} = T_{g_1g_2}$. $(G, K \times A)$ is isomorphic with $(G, G/N)$.

The natural action of A on $K \times A$, given by multiplication in the A coordinate $((k,a) \to (k,aa'))$, commutes with T_g, $g \in G$. Therefore, if Γ is any subgroup of G, a point $(k,a) \in K \times A$ has a dense Γ orbit if and only if all points (k,a'), $a' \in A$, have dense Γ orbits. In case Γ is discrete and cocompact, Moore's ergodicity theorem implies almost every point of G/Γ has a dense N orbit, and therefore there are points in $K \times A$ which have dense Γ orbits. Thus, there exists a point $k \in K$ such that for every $a \in A$, (k,a) has a dense Γ orbit.

3. Mostow's theorem

In the present section G is allowed to have compact factors and infinite center. Let $\mathfrak{G}$ be the Lie algebra of G. We recall that a subalgebra $\mathfrak{H} \subseteq \mathfrak{G}$ is a *Cartan subalgebra* if $\mathfrak{H}$ is a maximal abelian subalgebra of $\mathfrak{G}$, and if for each $X \in \mathfrak{H}$ $\mathrm{ad}_x(Y) = [X, Y]$ is a semisimple endomorphism of $\mathfrak{E}$ (i.e., diagonalizable over $\mathbf{C}$). If $\mathfrak{H}$ is a Cartan subalgebra of $\mathfrak{G}$, and if $H \subseteq G$ is the group of elements which in the adjoint representation leave $\mathfrak{H}$ pointwise fixed, H is called a *Cartan subgroup*. There exists a Cartan subgroup H such that $A \subseteq H \subseteq MA$.

THEOREM (Mostow [9]). *Let G be a connected semisimple Lie group, and let Γ be a discrete cocompact subgroup of G. If H is a Cartan subgroup of G, there exists $g \in G$ such that $g\Gamma g^{-1} \cap H$ is a cocompact subgroup of H.*

REMARK. Mostow's theorem has been extended by Prasad and Raghunathan to the case of discrete subgroups Γ for which G/Γ has finite volume ([11]).

4. Proof of Theorem 1.1

For the remainder of the paper G is a connected semi-simple Lie group with finite center and no nontrivial compact factors; Γ is a discrete cocompact subgroup of G, and H is a Cartan subgroup such that $A \subseteq H \subseteq MA$.

We claim it is enough to prove Theorem 1.1 under the additional assumption that $\Gamma \cap H$ is cocompact in H. For by Mostow's theorem there exists $g \in G$ such that $g\Gamma g^{-1}$ has this property, and if we known $g\Gamma g^{-1}hN$ is dense in G for all $h \in G$, it will be true that ΓxN is dense in G for all $x \in G$.

We remark that because $H \supseteq A$, $\Gamma \cap MA$ is cocompact in MA if $\Gamma \cap H$ is cocompact in H.

LEMMA 4.1. *With notations and assumptions as above, let $k' \in K$, $m' \in M$.*

There exist $m \in M$ *and a sequence* $\{\gamma_n\}$ *in* Γ *such that* (i) $\lim_{n\to\infty} k(\gamma_n m') = k'm$, *and* (ii) $\{a(\gamma_n m')\}$ *is a bounded sequence in* A.

PROOF. By the minimality of (Γ, K) there exists a sequence $\{\gamma'_n\}$ in Γ such that $\lim_{n\to\infty} k(\gamma'_n m') = k'$. Since $\Gamma \cap MA$ is cocompact in MA, there exists a sequence $\{\gamma''_n\}$ in $\Gamma \cap MA$ such that $a(\gamma'_n m')\gamma''_n$ is bounded in MA. Write $\gamma''_n = m''_n a''_n$, $m''_n \in M$, $a''_n \in A$. Setting $\gamma_n = \gamma'_n \gamma''_n$, we have

$$
\begin{aligned}
T_{\gamma_n}(m', e) &= (k(\gamma_n m'), \quad a(\gamma_n m')) \\
&= (k(\gamma'_n m''_n a''_n m'), \quad a(\gamma'_n m''_n a''_n m')) \qquad (4.2) \\
&= (k(\gamma'_n) m''_n m', \quad a(\gamma'_n) a''_n).
\end{aligned}
$$

Passing to a subsequence if necessary, we may suppose $m''_n m' \to m$ for some m, and so the first coordinate in (4.2) converges to $k'm$. Since $a(\gamma'_n) a''_n = a(\gamma'_n m')\gamma''_n (m''_n)^{-1}$, and since $\{a(\gamma'_n m')\gamma''_n\}$ is bounded by our choice of $\{\gamma''_n\}$, the sequence $\{a(\gamma_n m')\}$ is bounded in A. The lemma is proved.

LEMMA 4.3. *With notations and assumptions as above, if* $m' \in M$ *and* $a' \in A$, *the point* (m', a') *has a dense* Γ *orbit in* $K \times A$.

PROOF. There exists a point $k' \in K$ such that for all $a \in A$, (k', a) has a dense Γ orbit. Since the action of Γ on K commutes with the right action of M, $(k'm, a)$ has a dense Γ orbit for all $m \in M$, $a \in A$. Let $\{\gamma_n\}$ be a sequence satisfying (i) and (ii) of Lemma 4.1. Since $a(\gamma_n m')$ is bounded in A, we may choose a subsequence if necessary, and assume $\lim_{n\to\infty} a(\gamma_n m') = a$ exists. Thus, (m', e) has $(k'm, a)$ in its Γ orbit closure, and therefore (m', e) has a dense Γ orbit. It follows that (m', a) has a dense Γ orbit for all $a \in A$.

REMARK. The discussion so far applies equally well to discrete subgroups Γ such that G/Γ has finite volume. This is thanks to the Prasad-Raghunathan extension of Mostow's theorem mentioned earlier. The final stage of our argument will consist of proving that every point of $K \times A$ has some point (m, a) $(m \in M)$ in its Γ orbit closure. For this it is definitely necessary that Γ be cocompact.

Let $\boldsymbol{k}$, $\boldsymbol{a}$, $\boldsymbol{n}^+$ be the Lie algebras of K, A, N respectively, and let $\boldsymbol{a}^*$ be the (real) dual space to $\boldsymbol{a}$. An element $\lambda \in \boldsymbol{a}^*$ is a *root* if the space

$$\boldsymbol{g}^{\lambda} = \{X \in \boldsymbol{g} \mid [Y, X] = \lambda(Y)X,\ Y \in \boldsymbol{a}\}$$

has positive dimension. The set, Λ, of roots is finite, and there exists a linear ordering of $\boldsymbol{a}^*$ (compatible with the vector space structure) such that if

$\Lambda^+ = \{\lambda \in \Lambda \mid \lambda > 0\}$, then $\boldsymbol{n}^+ = \Sigma_{\lambda \in \Lambda^+} \boldsymbol{g}^\lambda$. If $\Lambda^- = \{\lambda \in \Lambda \mid \lambda < 0\}$, then $\boldsymbol{n}^- = \Sigma_{\lambda \in \Lambda^-} \boldsymbol{g}^\lambda$ is a Lie subalgebra of $\boldsymbol{g}$ and there is a direct sum decomposition $\boldsymbol{g} = \boldsymbol{n}^- + \boldsymbol{g}^0 + \boldsymbol{n}^+$, where $\boldsymbol{g}^0$ further decomposes as $\boldsymbol{g}^0 = \boldsymbol{m} + \boldsymbol{a}$, $\boldsymbol{m} \subseteq \boldsymbol{k}$, $\boldsymbol{m}$ the Lie algebra of the group M defined earlier. $N^- = \exp(\boldsymbol{n}^-)$ is diffeomorphic to an open dense subset of K/M under the map $x \to k(x)M$, $x \in N^-$. We note for later reference that $\mathrm{ad}_g N^- = N^-$ for $g \in M$ and $g \in A$. Also for $g \in A$, say $g = \exp(X)$, the derivative of ad_g in the direction of any vector $Y \in \boldsymbol{g}^\lambda$ is $\exp(\lambda(X))Y$. Therefore

$$\lim_{\substack{\lambda(X) \to -\infty \\ \lambda \in \Lambda \\ \exp(X) = g}} \mathrm{ad}_g h = e \tag{4.4}$$

uniformly on compact sets of h in N^-. For these facts and others cited below, see [12].

Let $B \subseteq N^-$ be a compact neighborhood of e. By the above $\mathcal{B} = \{k(b)M \mid b \in B\}$ is a compact neighborhood of eM in K/M. Now fix $k' \in K$, and define $\Gamma_{\mathcal{B}}(k') = \{\gamma \in \Gamma \mid k(\gamma k')M \in \mathcal{B}\}$. Since $(\Gamma, K/M)$ is *minimal*, $\Gamma_{\mathcal{B}}(k')$ is "left relatively dense." This means there exist $\gamma_1, \cdots, \gamma_p \in \Gamma$ such that $\gamma_1 \Gamma_{\mathcal{B}}(k') \cup \cdots \cup \gamma_p \Gamma_{\mathcal{B}}(k') = \Gamma$. Since Γ is cocompact, there exists a compact set $C' \subseteq G$ such that $C'\Gamma = G$, and so if we set $C = C'\gamma_1 \cup \cdots \cup C'\gamma_p$, C is compact, and $C\Gamma_{\mathcal{B}}(k') = G$.

Since $a: G \to A$ is continuous, the set $A_0 = \{a(gk) \mid g \in C, k \in K\}$ is compact in A. Define $A_{\mathcal{B}}(k') = \{(a(\gamma k') \mid \gamma \in \Gamma_{\mathcal{B}}(k')\}$. Given any $g \in G$, write $g = c\gamma$, $c \in C$, $\gamma \in \Gamma_{\mathcal{B}}(k')$. Then

$$\begin{aligned} a(gk') &= a(c\gamma k') \\ &= a(ck(\gamma k'))a(\gamma k') \\ &\in A_0 A_{\mathcal{B}}(k'). \end{aligned} \tag{4.5}$$

Denote by "log" the inverse to the exponential map of $\boldsymbol{a}$ onto A. We define two subsets of $\boldsymbol{a}$ by

$$\boldsymbol{a}_0 = \log A_0$$

$$\boldsymbol{a}_{\mathcal{B}}(k') = \log A_{\mathcal{B}}(k').$$

By (4.5) $\boldsymbol{a} = \boldsymbol{a}_0 + \boldsymbol{a}_{\mathcal{B}}(k')$.

Let $\|\cdot\|$ be a norm for $\boldsymbol{a}$. We use the dual norm on $\boldsymbol{a}^*$ and the same symbol. Let $\alpha = \max_{Y \in \boldsymbol{a}_0} \|Y\|$. Then for every $\lambda \in \boldsymbol{a}^*$ and $X \in \boldsymbol{a}$ there exists $X' \in \boldsymbol{a}_{\mathcal{B}}(k')$ such that $\|X - X'\| \leqq \alpha$ and $|\lambda(X) - \lambda(X')| \leqq \alpha \|\lambda\|$.

Let $0 < T < \infty$. T will be specified later. Choose $X \in \mathfrak{a}$ with $\lambda(X) > 3T$, $\lambda \in \Lambda^-$. There exists $X' \in \mathfrak{a}_{\mathscr{B}}(k')$ such that $\|X - X'\| \leqq \alpha$, and therefore $\lambda(X') > 3T - \|\lambda\|\alpha$, $\lambda \in \Lambda^-$. Assume now that $T > |\lambda\|\alpha$, $\lambda \in \Lambda^-$. We have

$$\lambda(X') > 2T \qquad (\lambda \in \Lambda^-). \tag{4.6}$$

By definition there exists $\gamma \in \Gamma$ such that $k(\gamma k')M \in \mathscr{B}$ and $\log a(\gamma k') = X'$. Because $\Gamma \cap MA$ is cocompact in MA, there exists $D = D(\Gamma) < \infty$ such that $\Gamma \cap MA$ contains some $\delta = m_1 a_1$ with $\|\log a_1 + X'\| \leqq D$. If, as we now assume, $T > D\|\lambda\|$, $\lambda \in \Lambda^-$, and if $X_1 = \log a_1$, then

$$\lambda(X_1) < -T \qquad (\lambda \in \Lambda^-). \tag{4.7}$$

Now let U be an arbitrary neigborhood of e in N^-. We shall prove the point (k', e) has in its Γ orbit a point of the form $(mk(u), a(u)a_2)$ for some $u \in U$ and $a_2 \in A$ with $\|\log a_2\| \leqq 2D$. Letting U decrease to the identity, it follows that the orbit closure of (k', e) contains a point (m, a), $m \in M$ ($\|\log a\| \leqq 2D$), and therefore (k', e) has a dense orbit. This is all that is needed to complete the proof of Theorem 1.1.

Define $B_0 \subseteq N^-$ by $B_0 = \{mbm^{-1} | m \in M,\ b \in B\}$. (Recall that B was a fixed compact neighborhood of e in N^-.) By (4.4) there exists T_0 such that if $T \geqq T_0$, and if (4.7) holds, then

$$a_1 B_0 a_1^{-1} \subseteq U. \tag{4.8}$$

We assume $T \geqq T_0$ (as well as $T \geqq \|\lambda\|\alpha$, $T \geqq D\|\lambda\|$, $\lambda \in \Lambda^-$).

We have selected two elements of Γ. The first, γ, has the property that $\gamma \in \Gamma_{\mathscr{B}}(k')$, and $\log a(\gamma k') = X'$ satisfies (4.6). The second, $\delta = m_1 a_1$, satisfies $\|\log a_1 + X'\| \leqq D$, and $X_1 = \log a_1$ satisfies (4.7). It will develop that $T_\delta T_\gamma(k', e)$ has the desired form $(mk(u), a(u)a_2)$, $\|\log a_2\| \leqq 2D$, $u \in U$, provided, as we now assume, $\|\log a(b)\| \leqq D$, $b \in B$.

To say that $\delta \in \Gamma_{\mathscr{B}}(k')$ is to say there exist $m_2 \in M$, $b \in B$, and $b_0 = m_2^{-1} b m_2 \in B_0$ such that

$$\begin{aligned} k(\gamma k') &= k(b) m_2 \\ &= m_2 m_2^{-1} k(b) m_2 \\ &= m_2 k(m_2^{-1} b m_2) \\ &= m_2 k(b_0). \end{aligned} \tag{4.9}$$

By (4.8), $u = a_1 b_0 a_1^{-1} \in U$. Using (4.9) we calculate $k(\delta\gamma k')$:

$$k(\delta\gamma k') = k(m_1 a_1 m_2 k(b_0))$$

$$= m_1 m_2 k(a_1 k(b_0))$$

(4.10)
$$= m_1 m_2 k(a_1 b_0)$$

$$= m_1 m_2 k(a_1 b_0 a_1^{-1})$$

$$= mk(u) \quad (m = m_1 m_2).$$

Let $a_2 = a(b_0)^{-1} a_1 a(\gamma k')$. We have from above $\|\log a_2\| = \|X_1 + X'\| \leqq 2D$. Using (4.9) once more,

$$a(\delta\gamma k') = a(m_1 a_1 k(\gamma k' k)) a(\gamma k')$$

$$= a(a_1 k(b_0)) a(\gamma k')$$

(4.11)
$$= a(a_1 b_0 a_1^{-1}) a(b_0)^{-1} a_1 a(\gamma k')$$

$$= a(u) a_2 .$$

Collecting results, (4.10) and (4.11) imply

$$T_\delta T_\gamma (k', e) = (mk(u),\ a(u)a_2)$$

as desired. Theorem 1.1 is proved.

Note Added in Proof. Since this paper was written we have established Theorem 1.1 with "minimal" replaced by "uniquely ergodic".

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